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ORIGINAL PAPER

Optimality conditions and duality for a class of nondifferentiable multi-objective fractional programming problems

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Abstract A class of multi-objective fractional programming problems (MFP) are considered where the involved functions are locally Lipschitz. In order to deduce our main results, we give the definition of the generalized (F, θ, ρ, d)-convex class about the Clarke's generalized gradient. Under the above generalized convexity assumption, necessary and sufficient conditions for optimality are given. Finally, a dual problem corresponding to (MFP) is formulated, appropriate dual theorems are proved.

Keywords Multi-objective fractional programming · Sublinear functions · Generalized convex functions · Optimality conditions · Duality

AMS Subject Classification 90C29, 90C32, 90C46

1 Introduction

Convexity plays a very important role in optimization theory. But for many mathematical models used in decision sciences, economics, management sciences, stochastics, applied mathematics and engineering, the notion of convexity does no longer suffice. To relax convexity assumptions imposed on the functions in theorems on optimality and duality, several definitions extending the concept of convexity of a function have been introduced. Hanson [1] introduced the concept of invexity, generalizing the difference $x - y$ in the definition of convex function to any function $\eta(x, y)$. He established Karush–Kuhn–Tucker type sufficient optimality conditions for the scalar

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optimization problem. During the last 20 years, numerous articles have appeared in the literature reflecting further generalizations and applications in this category (see, e.g. [2–10]). Under various kinds of generalized convexities, some results for optimality conditions and duality in multi-objective fractional programming problems (MFP) have been obtained (see [11–20]). In particular, Bector et al. [11] derived Fritz John and Karush–Kuhn–Tucker necessary and sufficient optimality conditions for a class of nondifferentiable convex (MFP), and they established some duality theorems. Following the approaches of Bector et al. [11], Liu [14,15] obtained necessary and sufficient conditions and derived duality theorem for a class of nonsmooth (MFP) involving either pseudoinvex functions or (F,ρ)-convex. In a recent paper [20], Liang et al. introduced the concept of differentiable (F,θ,ρ,d)-convexity, and he proved optimality theorems and duality results for a (MFP) involving (F,θ,ρ,d)-convexity functions.

In this paper, we consider the following (MFP):

$$\text{(MFP)} \qquad \min \left(\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right)$$
$$\text{s.t.} \quad h_j(x)\le 0,\ \ j=1,\ldots,m,$$

where $f_i\colon R^n \to R,\ \ g_i\colon R^n \to R,\ \ i=1,\ldots,p,\ \ h_j\colon R^n \to R,\ j=1,\ldots,m,$ are locally Lipschitz and $\frac{f_i}{g_i}: R^n \to R,\ i=1,\ldots,m,$ are locally Lipschitz too. And suppose $f_i(x)\ge 0,\ \ g_i(x)>0,\ \ \forall x\in R^n$. Denote by S the set of all feasible solutions for (MFP), i.e., $S=\{x\in R^n|h_j(x)\le 0, j=1,\ldots,m\}$.

For (MFP), we introduced a new concept of generalized (F,θ,ρ,d)-convexity about the Clarke's generalized gradient at first. Then optimality conditions are obtained. Finally a dual model is formulated and duality results are proved using the concept of generalized (F,θ,ρ,d)-convexity.

The organization of the remainder of this paper is as follows. In Sect. 2, some definitions and notations are given. In Sect. 3, we establish necessary and sufficient conditions for (MFP) involving generalized (F,θ,ρ,d)-convex maps. Finally, duality theorems are presented in Sect. 4.

2 Preliminaries and definitions

Throughout this paper, let R^n be the n-dimensional Euclidean space and R^n_+ be nonnegative orthant of R^n. The following convention for any vectors $x,\ \ y\in R^n$ will be adopted:

$$x>y \Longleftrightarrow x_i>y_i \ \text{ for all } i=1,\ldots,n,$$

$$x\ge y \Longleftrightarrow x_i\ge y_i \ \text{ for all } i=1,\ldots,n.$$

Definition 1 A feasible solution $u\in S$ of (MFP) is called an efficient solution of (MFP) if there exists no other feasible solution $x\in S$ such that

$$\frac{f_i(x)}{g_i(x)}\le\frac{f_i(u)}{g_i(u)}, \quad \text{for all } i=1,\ldots,p$$

and at least one inequality holds strictly.

Definition 2 [21]. The real-valued function $f: R^n \to R$ is said to be locally Lipschitz if for any $z \in R^n$ there exists a positive constant k and a neighborhood N of z such that, for each $x, y \in N$,

$$|f(x) - f(y)| \leq k\|x - y\|,$$

where $\|\cdot\|$ denotes any norm in R^n.

For each d in R^n, $f^{\circ}(x;d)$ is the Clarke's generalized directional derivative [21] of a locally Lipschitz function f at x in the direction d defined by

$$f^{\circ}(x;d) = \limsup_{\substack{y \to x \\ t \to 0^+}} t^{-1}\{f(y + td) - f(y)\}.$$

It then follows that for any x and d

$$f^{\circ}(x;d) = \max\{\xi^T d | \xi \in \partial f(x)\},$$

where $\partial f(\cdot)$ denotes the Clarke's generalized gradient [21].

Lemma 1 [21]. *Let φ_1, φ_2: $R^n \to R$ be Lipschitz near x. If $\varphi_1(x) \geq 0$, $\varphi_2(x) > 0$, and if φ_1, $-\varphi_2$ are regular at x, then*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(x) = \frac{\varphi_2(x)\partial\varphi_1(x) - \varphi_1(x)\partial\varphi_2(x)}{\varphi_2^2(x)}.$$

Definition 3 [22]. A function F: $R^n \times R^n \times R^n \to R$ is called sublinear if, for any $x_1, x_2 \in R^n$,

$$F(x_1, x_2; \alpha_1 + \alpha_2) \leq F(x_1, x_2; \alpha_1) + F(x_1, x_2; \alpha_2), \quad \forall \alpha_1, \alpha_2 \in R^n, \tag{1}$$

$$F(x_1, x_2; r\alpha) = rF(x_1, x_2; \alpha), \quad \forall r \in R, r \geq 0, \alpha \in R^n. \tag{2}$$

It follows from (2) that

$$F(x_1, x_2; 0) = F(x_1, x_2; 0\alpha) = 0F(x_1, x_2; \alpha) = 0, \quad \forall \alpha \in R^n.$$

Definition 4 Let F: $R^n \times R^n \times R^n \to R$ be a sublinear function; let function $\varphi: R^n \to R$ be locally Lipschitz at $x^0 \in R^n$, $\theta: R^n \times R^n \to R_+\backslash\{0\}$, $\rho \in R$, and $d: R^n \times R^n \to R$. The function φ is said to be generalized (F, θ, ρ, d)-convex at x^0 if for all $\xi \in \partial\varphi(x^0)$ and for all $x \in R^n$, we have

$$\varphi(x) - \varphi(x^0) \geq F\left(x, x^0; \theta(x, x^0)\xi\right) + \rho d^2(x, x^0).$$

The function φ is called to be generalized (F, θ, ρ, d)-convex on R^n, if it is generalized (F, θ, ρ, d)-convex at every point in R^n.

Remark

(1) If $F\left(x, x^0; \theta(x, x^0)\xi\right) = \xi^T\eta(x, x^0)$ in the above definition, then we get ρ-invexity [23].
(2) If φ is continuous differentiable at x^0, then we obtain (F, θ, ρ, d)-type convexity [2].
(3) If $\theta(x, x^0)$=1 in the above definitions, then we get generalized (F, ρ)-convexity [19].

Theorem 1 *Assume $f : R^n \to R$, $g : R^n \to R$, are locally Lipschitz functions and $f(x) \geq 0$, $g(x) > 0$ for all $x \in R^n$. If f and $-g$ are generalized (F, θ, ρ, d)-convex at x^0, and f, $-g$ are regular at x^0, then $\frac{f}{g}$ is generalized $(F, \bar{\theta}, \bar{\rho}, \bar{d})$-convex at x^0, where*

$$\bar{\theta} = \frac{\theta(x, x^0) g(x^0)}{g(x)}, \qquad \bar{\rho} = \rho\left(1 + \frac{f(x^0)}{g(x^0)}\right), \qquad \bar{d}(x, x^0) = \frac{d(x, x^0)}{g^{\frac{1}{2}}(x)}.$$

Proof For any $x \in R^n$, we only need to prove that $\forall \eta \in \partial\left(\frac{f}{g}\right)(x^0)$, the following

$$\frac{f(x)}{g(x)} - \frac{f(x^0)}{g(x^0)} \geq F\left(x, x^0; \bar{\theta}(x, x^0)\eta\right) + \rho\bar{d}^2(x, x^0)$$

holds. For any $\eta \in \partial\left(\frac{f}{g}\right)(x^0)$, by Lemma 1, there exist $\alpha \in \partial f(x^0), \beta \in \partial g(x^0)$ such that

$$\eta = \frac{g(x^0)\alpha - f(x^0)\beta}{g^2(x^0)}. \tag{3}$$

By the generalized (F, θ, ρ, d)-convexity of f and $-g$ at $x^0, f \geq 0, g > 0$, we have that

$$\begin{aligned}
\frac{f(x)}{g(x)} - \frac{f(x^0)}{g(x^0)} &= \frac{1}{g(x)}\left(f(x) - f(x^0)\right) - \frac{f(x^0)}{g(x)g(x^0)}\left(g(x) - g(x^0)\right) \\
&\geq \frac{1}{g(x)}\left[F(x, x^0; \theta(x, x^0)\alpha) + \rho d^2(x, x^0)\right] \\
&\quad + \frac{f(x^0)}{g(x)g(x^0)}\left[F(x, x^0; -\theta(x, x^0)\beta) + \rho d^2(x, x^0)\right] \\
&= \frac{g(x^0)}{g(x)}\left[F\left(x, x^0; \frac{\theta(x, x^0)\alpha}{g(x^0)}\right) + \frac{\rho}{g(x^0)} d^2(x, x^0)\right] \\
&\quad + \frac{g(x^0)}{g(x)}\left[F\left(x, x^0; -\frac{\theta(x, x^0)\beta f(x^0)}{g^2(x^0)}\right) + \frac{f(x^0)\rho}{g^2(x^0)} d^2(x, x^0)\right].
\end{aligned}$$

Using (1), then we have

$$\begin{aligned}
\frac{f(x)}{g(x)} - \frac{f(x^0)}{g(x^0)} &\geq \frac{g(x^0)}{g(x)} F\left(x, x^0; \frac{\theta(x, x^0)\alpha}{g(x^0)} - \frac{\theta(x, x^0)\beta f(x^0)}{g^2(x^0)}\right) + \rho\left[\frac{d(x, x^0)}{g^{\frac{1}{2}}(x)}\right]^2 \\
&\quad + \rho\left[\left(\frac{f(x^0)}{g(x)g(x^0)}\right)^{\frac{1}{2}} d(x, x^0)\right]^2 \\
&= \frac{g(x^0)}{g(x)} F\left(x, x^0; \theta(x, x^0)\frac{\alpha g(x^0) - \beta f(x^0)}{g^2(x^0)}\right) \\
&\quad + \rho\frac{1}{g(x)}\left(1 + \frac{f(x^0)}{g(x^0)}\right) d^2(x, x^0).
\end{aligned}$$

By (2) and (3), we obtain

$$\frac{f(x)}{g(x)} - \frac{f(x^0)}{g(x^0)} \geq F\left(x, x^0; \theta(x, x^0)\eta \frac{g(x^0)}{g(x)}\right) + \rho \frac{1}{g(x)}\left(1 + \frac{f(x^0)}{g(x^0)}\right) d^2(x, x^0).$$

Therefore, the function $\frac{f}{g}$ is generalized $(F, \bar{\theta}, \rho, \bar{d})$-convex at x^0, where

$$\bar{\theta}(x, x^0) = \frac{g(x^0)}{g(x)}\theta(x, x^0) > 0, \qquad \bar{\rho} = \rho(1 + \frac{f(x^0)}{g(x^0)}), \qquad \bar{d}(x, x^0) = \left(\frac{1}{g(x)}\right)^{1/2} d(x, x^0).$$

3 Optimality conditions

In this section, we establish generalized Karush–Kuhn–Tucker necessary and sufficient optimality conditions for efficient solutions of (MFP).

Consider the following scalar minimization problem:

$$\text{(P)} \qquad \begin{array}{l} \min \quad q(x) \\ \text{s.t. } h_j(x) \leq 0, \quad j = 1, \ldots, m, \end{array}$$

where $q, h_j : R^n \to R$, $j = 1, \ldots, m$, are locally Lipschitz functions.

At a point $u \in R^n$, let us define

$$J^* = \{j \in J | h_j(u) = 0\}, \quad J = \{1, \ldots, m\},$$

$$\Omega^\circ = \begin{cases} \{d \in R^n | h_j^\circ(u; d) \leq 0, j \in J^*\}, & \text{if } J^* \neq \emptyset, \\ R^n, & \text{if } J^* = \emptyset, \end{cases}$$

$$\Omega_-^\circ = \begin{cases} \{d \in R^n | h_j^\circ(u; d) < 0, j \in J^*\}, & \text{if } J^* \neq \emptyset, \\ R^n, & \text{if } J^* = \emptyset. \end{cases}$$

For problem (P), we assume the following constraint qualification.

Constraint Qualification A. At a point u, it holds that $\Omega_-^\circ \neq \emptyset$.

The following result is a well-known necessary optimality condition, which can be found in [24].

Lemma 2 [24]. *If u is a local minimum for* (P) *and Constraint Qualification A is satisfied, then there exist $v_1, \ldots, v_m$ such that*

$$0 \in \partial q(u) + \sum_{j=1}^{m} v_j \partial h_j(u),$$

$$v_j h_j(u) = 0, \quad j = 1, \ldots, m,$$

$$v_j \geq 0, \quad j = 1, \ldots, m.$$

Lemma 3 [25]. *u is an efficient solution for* (MFP) *if and only if u solves* ($\mathrm{FP_k}$)*, $k = 1, \ldots, p$, where* ($\mathrm{FP_k}$) *is the following problems:*

$$(\mathrm{FP_k}) \qquad \begin{array}{ll} \min & \dfrac{f_k(x)}{g_k(x)} \\ s.t. & \dfrac{f_i(x)}{g_i(x)} \leq \dfrac{f_i(u)}{g_i(u)} \quad \text{for all } i = 1, \ldots, p \text{ but } i \neq k, \\ & h_j(x) \leq 0, \quad j = 1, \ldots, m. \end{array}$$

We can prove the following generalized Karush–Kuhn–Tucher type necessary optimality theorem for (MFP).

Theorem 2 (Necessary Optimality Conditions) *If u is an efficient solution for* (MFP) *and satisfies Constraint Qualification A for* ($\mathrm{FP_k}$)*, $k = 1, \ldots, p$, then there exist $\tau \in R^p$ and $\lambda \in R^m$ such that*

$$0 \in \sum_{i=1}^{p} \tau_i \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j \partial h_j(u),$$

$$\lambda_j h_j(u) = 0, \quad j = 1, \ldots, m,$$

$$\tau = (\tau_1, \ldots, \tau_p) > 0, \qquad \lambda = (\lambda_1, \ldots, \lambda_m) \geq 0.$$

Proof Since u is an efficient solution for (MFP), by Lemma 3, for any $k \in \{1, \ldots, p\}$, u is a minimal solution of ($\mathrm{FP_k}$), where

$$(\mathrm{FP_k}) \qquad \begin{array}{ll} \min & \dfrac{f_k(x)}{g_k(x)} \\ \text{s.t.} & \dfrac{f_i(x)}{g_i(x)} \leq \dfrac{f_i(u)}{g_i(u)} \quad \text{for all } i = 1, \ldots, p \text{ but } i \neq k, \\ & h_j(x) \leq 0, \quad j = 1, \ldots, m. \end{array}$$

Since u satisfies Constraint Qualification A for ($\mathrm{FP_k}$), by Lemma 2, there exist $\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}, t_1^{(k)}, \ldots, t_{k-1}^{(k)}, t_{k+1}^{(k)} \ldots, t_p^{(k)}$, such that

$$0 \in \partial \left(\frac{f_k}{g_k} \right)(u) + \sum_{\substack{i=1 \\ i \neq k}}^{p} t_i^{(k)} \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j^{(k)} \partial h_j(u), \tag{4}$$

$$\lambda_j^{(k)} h_j(u) = 0, \quad j = 1, \ldots, m, \tag{5}$$

$$\lambda_j^{(k)} \geq 0, \quad j = 1, \ldots, m, \tag{6}$$

$$t_i^{(k)} \geq 0, \quad i = 1, \ldots, k-1, \; k+1, \ldots, p. \tag{7}$$

Let $k = 1$ in (4)–(7), we obtain

$$0 \in \partial \left(\frac{f_1}{g_1} \right)(u) + \sum_{i=2}^{p} t_i^{(1)} \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j^{(1)} \partial h_j(u),$$

$$\lambda_j^{(1)} h_j(u) = 0, \quad j = 1, \dots, m,$$

$$\lambda_j^{(1)} \geq 0, \quad j = 1, \dots, m,$$

$$t_i^{(1)} \geq 0, \quad i = 1, \dots, p.$$

Let $k = 2$ in (4)–(7), we obtain

$$0 \in \partial \left(\frac{f_2}{g_2} \right)(u) + \sum_{\substack{i=1 \\ i \neq 2}}^{p} t_i^{(2)} \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j^{(2)} \partial h_j(u),$$

$$\lambda_j^{(2)} h_j(u) = 0, \quad j = 1, \dots, m,$$

$$\lambda_j^{(2)} \geq 0, \quad j = 1, \dots, m,$$

$$t_i^{(2)} \geq 0, \quad i = 1, 3, 4, \dots, p.$$

Similarly, let $k = p$ in (4)–(7), we obtain

$$0 \in \partial \left(\frac{f_p}{g_p} \right)(u) + \sum_{i=1}^{p-1} t_i^{(p)} \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j^{(p)} \partial h_j(u),$$

$$\lambda_j^{(p)} h_j(u) = 0, \quad j = 1, \dots, m,$$

$$\lambda_j^{(p)} \geq 0, \quad j = 1, \dots, m,$$

$$t_i^{(p)} \geq 0, \quad i = 1, \dots, p-1.$$

Since $\partial \left(\frac{f}{g} \right)(u)$, $\partial \left(\frac{f_i}{g_i} \right)(u)$, $i = 1, \dots, p$, are convex, according to the above discussion, we have

$$0 \in \sum_{i=1}^{p} \tau_i \partial \left(\frac{f_i}{g_i} \right)(u) + \sum_{j=1}^{m} \lambda_j \partial h_j(u),$$

$$\lambda_j h_j(u) = 0, \quad j = 1, \dots, m,$$

where $\tau_i = t_i^{(1)} + t_i^{(2)} + \cdots + t_i^{(i-1)} + 1 + t_i^{(i+1)} + \cdots + t_i^{(p)} \geq 1, \lambda_j = \sum_{k=1}^{p} \lambda_j^{(k)} \geq 0$. The proof is complete.

In the following, we present some sufficient efficiency conditions for (MFP) under the generalized (F, θ, ρ, d)-convexity assumptions.

Theorem 3 (Sufficient Optimality Conditions) *Let x^0 be a feasible solution of* (MFP). *Suppose that there exist $\tau = (\tau_1, \ldots, \tau_p) \in R^p$, $\tau > 0$, and $\lambda = (\lambda_1, \ldots, \lambda_m) \in R^m_+$ such that*

$$0 \in \sum_{i=1}^{p} \tau_i \partial \left(\frac{f_i}{g_i}\right)(x^0) + \sum_{j=1}^{m} \lambda_j \partial h_j(x^0), \tag{8}$$

$$\lambda_j h_j(x^0) = 0, \quad j = 1, \ldots, m. \tag{9}$$

If f_i and $-g_i$ $(i = 1, \ldots, p)$ are generalized $(F, \theta_{1i}, \rho_{1i}, d_{1i})$-convex at x^0, $h_j (j = 1, \ldots, m)$ are generalized $(F, \theta_{2j}, \rho_{2j}, d_{2j})$-convex at x^0, and

$$\sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x, x^0)]^2}{\bar{\theta}_i(x, x^0)} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x, x^0)]^2}{\theta_{2j}(x, x^0)} \geq 0, \tag{10}$$

where $\bar{\theta}_i(x, x^0) = \dfrac{\theta_{1i}(x, x^0) g_i(x^0)}{g_i(x)}$, $\bar{\rho}_i = \rho_{1i}\left(1 + \dfrac{f_i(x^0)}{g_i(x^0)}\right)$, *and* $\bar{d}_i(x, x^0) = \dfrac{d_{1i}(x, x^0)}{g_i^{\frac{1}{2}}(x)}$. *Furthermore, if f_i and $-g_i$ $(i = 1, \ldots, p)$ are regular at x^0, then x^0 is an efficient solution for* (MFP).

Proof By (8), there exist $\eta_i' \in \partial \left(\frac{f_i}{g_i}\right)(x^0)$, $i = 1, \ldots, p$, $\gamma_j' \in \partial h_j(x^0)$, such that

$$\sum_{i=1}^{p} \tau_i \eta_i' + \sum_{j=1}^{m} \lambda_j \gamma_j' = 0. \tag{11}$$

Suppose that x^0 is not an efficient solution of (MFP). Then there exists a feasible solution x of (MFP) such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^0)}{g_i(x^0)}, \quad i = 1, \ldots, p \tag{12}$$

and at least one inequality holds strictly.

By Theorem 1, for each i, $1 \leq i \leq p$, $\dfrac{f_i}{g_i}$ is generalized $(F, \bar{\theta}_i, \bar{\rho}_i, \bar{d}_i)$-convex at x^0, we have

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^0)}{g_i(x^0)} \geq F(x, x^0; \bar{\theta}_i(x, x^0)\eta_i') + \bar{\rho}_i [\bar{d}_i(x, x^0)]^2,$$

where $\bar{\theta}_i(x, x^0) = \dfrac{\theta_{1i}(x, x^0) g_i(x^0)}{g_i(x)}$, $\bar{\rho}_i = \rho_{1i}\left(1 + \dfrac{f_i(x^0)}{g_i(x^0)}\right)$, and $\bar{d}_i(x, x^0) = \dfrac{d_{1i}(x, x^0)}{g_i^{\frac{1}{2}}(x)}$.

Since $\bar{\theta}_i(x, x^0) > 0$, by the sublinearity of F and (12), we have

$$0 \geq \frac{1}{\bar{\theta}_i(x, x^0)} \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^0)}{g_i(x^0)}\right) \geq F(x, x^0; \eta_i') + \bar{\rho}_i \frac{[\bar{d}_i(x, x^0)]^2}{\bar{\theta}_i(x, x^0)}.$$

Hence, we obtain the following p inequalities

$$F(x, x^0; \eta_i') + \bar{\rho}_i \frac{[\bar{d}_i(x, x^0)]^2}{\bar{\theta}_i(x, x^0)} \leq 0, \quad i = 1, \ldots, p$$

and at least one inequality holds strictly.

Multiplying the above p inequalities with τ_i, respectively, and then adding them together, we have

$$\sum_{i=1}^{p} \tau_i F(x, x^0; \eta_i') + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} < 0.$$

By the sublinearity of F and $\tau_i > 0$, $i = 1, \ldots, p$, we have that

$$\sum_{i=1}^{p} \tau_i F\left(x, x^0; \eta_i'\right) \geq F(x, x^0; \sum_{i=1}^{p} \tau_i \eta_i').$$

Hence, we get

$$F\left(x, x^0; \sum_{i=1}^{p} \tau_i \eta_i'\right) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} < 0. \tag{13}$$

Substituting (11) into (13), we obtain

$$F\left(x, x^0; -\sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} < 0. \tag{14}$$

The sublinearity of F and (10) yield

$$\begin{aligned}
&F\left(x, x^0; -\sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} \\
&\quad + F\left(x, x^0; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)} \\
&\geq F(x, x^0; 0) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)} \\
&= \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{\left[\bar{d}_i(x, x^0)\right]^2}{\bar{\theta}_i(x, x^0)} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)} \\
&\geq 0.
\end{aligned}$$

Using (14), we obtain

$$F\left(x, x^0; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)} > 0. \tag{15}$$

On the other hand, for $j = 1, \ldots, m$, by the generalized $(F, \theta_{2j}, \rho_{2j}, d_{2j})$-convexity of h_j at x^0, we have

$$h_j(x) - h_j(x^0) \geq F\left(x, x^0; \theta_{2j}(x, x^0)\gamma_j'\right) + \rho_{2j}[d_{2j}(x, x^0)]^2.$$

By using $\lambda_j \geq 0, \theta_{2j}(x, x^0) > 0, j = 1, \ldots, m$ and the sublinearity of F, we have

$$\sum_{j=1}^{m} \lambda_j \frac{h_j(x) - h_j(x^0)}{\theta_{2j}(x, x^0)} \geq F\left(x, x^0; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)}.$$

Since x is feasible for (MFP) and $\theta_{2j}(x, x^0) > 0$, (9) implies that

$$\sum_{j=1}^{m} \lambda_j \frac{h_j(x) - h_j(x^0)}{\theta_{2j}(x, x^0)} \leq 0.$$

Then, we obtain

$$F\left(x, x^0; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{\left[d_{2j}(x, x^0)\right]^2}{\theta_{2j}(x, x^0)} \leq 0, \tag{16}$$

which contradicts (15). Therefore, x^0 is an efficient solution for (MFP). The proof is complete.

4 Duality

In this section, we formulate the following dual problem for (MFP):

$$\text{(MFD)} \quad \begin{array}{ll} \max & \left(\dfrac{f_1(u)}{g_1(u)}, \ldots, \dfrac{f_p(u)}{g_p(u)}\right) \\ \text{s.t.} & 0 \in \sum_{i=1}^{p} \tau_i \partial \left(\dfrac{f_i}{g_i}\right)(u) + \sum_{j=1}^{m} \lambda_j \partial h_j(u) \\ & \lambda_j h_j(u) \geq 0, \quad j = 1, \ldots, m, \\ & \tau = (\tau_1, \ldots, \tau_p) \in R^p, \ \ \tau > 0, \\ & \lambda = (\lambda_1, \ldots, \lambda_m) \in R_+^m. \end{array}$$

We establish weak, strong duality theorems between (MFP) and (MFD).

Theorem 4 *Let x be a feasible solution for* (MFP) *and let (u, τ, λ) be a feasible solution for* (MFD). *Suppose f_i and $-g_i$ are regular at u. If f_i and $-g_i$ $(i = 1, \ldots, p)$ are generalized $(F, \theta_{1i}, \rho_{1i}, d_{1i})$-convex at u, h_j $(j = 1, \ldots, m)$ are generalized $(F, \theta_{2j}, \rho_{2j}, d_{2j})$-convex at u, and the inequality*

$$\sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x, u)]^2}{\bar{\theta}_i(x, u)} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x, u)]^2}{\theta_{2j}(x, u)} \geq 0 \tag{17}$$

holds, where $\bar{\theta}_i(x, u) = \theta_{1i}(x, u) \dfrac{g_i(u)}{g_i(x)}$, $\bar{\rho}_i = \rho_{1i}\left(1 + \dfrac{f_i(u)}{g_i(u)}\right)$ *and* $\bar{d}_i(x, u) = \dfrac{d_{1i}(x, u)}{[g_i(x)]^{1/2}}$, *then*

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u)}{g_i(u)}, \quad \text{for all } i = 1, \ldots, p$$

and

$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(u)}{g_k(u)}, \quad \textit{for some } k \in \{1, \ldots, p\}$$

can not hold.

Proof Since that (u, τ, λ) is a feasible solution for (MFD), we have

$$0 \in \sum_{i=1}^{p} \tau_i \partial \left(\frac{f_i}{g_i}\right)(u) + \sum_{j=1}^{m} \lambda_j \partial h_j(u), \tag{18}$$

$$\lambda_j h_j(u) \geq 0, \quad j = 1, \ldots, m, \tag{19}$$

$$\tau = (\tau_1, \ldots, \tau_p) > 0, \quad \lambda = (\lambda_1, \ldots, \lambda_m) \geq 0. \tag{20}$$

Therefore, there exist $\eta_i' \in \partial \left(\frac{f_i}{g_i}\right)(u)$, $i = 1, \ldots, p$, $\gamma_j' \in \partial h_j(u)$, $j = 1, \ldots, m$, such that

$$\sum_{i=1}^{p} \tau_i \eta_i' + \sum_{j=1}^{m} \lambda_j \gamma_j' = 0. \tag{21}$$

Suppose, contrary to the result of the theorem, that for a feasible solution x for (MFP) and a feasible solution (u, τ, λ) for (MFD)

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \leq 0, \quad i = 1, \ldots, p \tag{22}$$

and at least one inequality holds strictly. For each i, $1 \leq i \leq p$, by the generalized convexity assumptions and Theorem 1, we have

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \geq F\left(x, u; \bar{\theta}_i(x, u)\eta_i'\right) + \bar{\rho}_i[\bar{d}_i(x, u)]^2,$$

where $\bar{\theta}_i(x, u) = \frac{\theta_{1i}(x, u)g_i(u)}{g_i(x)}$, $\bar{\rho}_i = \rho_{1i}\left(1 + \frac{f_i(u)}{g_i(u)}\right)$, and $\bar{d}_i(x, u) = \frac{d_{1i}(x, u)}{g_i^{1/2}(x)}$. Using $\tau_i > 0, \bar{\theta}_i(x, u) > 0$ and (2), we get

$$\frac{\tau_i}{\bar{\theta}_i(x, u)}\left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)}\right) \geq F(x, u; \tau_i \eta_i') + \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x, u)]^2}{\bar{\theta}_i(x, u)}.$$

Then, by (22), we obtain

$$F(x, u; \tau_i \eta_i') + \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x, u)]^2}{\bar{\theta}_i(x, u)} \leq 0, \quad i = 1, \ldots, p.$$

Furthermore, at least one of the above inequalities holds strictly. After adding these inequalities together, we get

$$\sum_{i=1}^{p} F(x, u; \tau_i \eta_i') + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x, u)]^2}{\bar{\theta}_i(x, u)} < 0.$$

Hence, it follows from (1) that

$$F\left(x, u; \sum_{i=1}^{p} \tau_i \eta_i'\right) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x,u)]^2}{\bar{\theta}_i(x,u)} < 0. \tag{23}$$

By the $(F, \theta_{2j}, \rho_{2j}, d_{2j})$-convexity of h_j at $u, j = 1, \ldots, m$, we have

$$h_j(x) - h_j(u) \geq F(x, u; \theta_{2j}(x,u)\gamma_j') + \rho_{2j}[d_{2j}(x,u)]^2.$$

Using $\lambda_j \geq 0$ and $\theta_{2j}(x,u) > 0$, we get

$$\lambda_j \frac{h_j(x) - h_j(u)}{\theta_{2j}(x,u)} \geq F(x, u; \lambda_j \gamma_j') + \lambda_j \rho_{2j} \frac{[d_{2j}(x,u)]^2}{\theta_{2j}(x,u)}, \quad j = 1, \ldots, m.$$

Adding these inequalities together and using the feasibility of x and (u, τ, λ), we obtain

$$\sum_{j=1}^{m} F(x, u; \lambda_j \gamma_j') + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x,u)]^2}{\theta_{2j}(x,u)} \leq 0.$$

Using (1) again, we have

$$F\left(x, u; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x,u)]^2}{\theta_{2j}(x,u)} \leq 0. \tag{24}$$

By (23) and (24), we have

$$F\left(x, u; \sum_{i=1}^{p} \tau_i \eta_i'\right) + \sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x,u)]^2}{\bar{\theta}_i(x,u)} + F\left(x, u; \sum_{j=1}^{m} \lambda_j \gamma_j'\right) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x,u)]^2}{\theta_{2j}(x,u)} < 0.$$

Based on the sublinearity of F, by (21), we obtain

$$\sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x,u)]^2}{\bar{\theta}_i(x,u)} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x,u)]^2}{\theta_{2j}(x,u)} < 0.$$

which contradicts (17). The proof is complete.

Theorem 5 *Assume that $\bar{x}$ is an efficient solution of* (MFP) *and the Constraint Qualification A holds at $\bar{x}$ for* (FP_k), $k = 1, \ldots, p$. *Then there exist $\bar{\tau} \in R^p$, $\bar{\lambda} \in R^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a feasible solution for* (MFD). *Furthermore, suppose $f_i, -g_i$, $i = 1, \ldots, p$ are regular at $\bar{x}$. If $f_i, -g_i, i = 1, \ldots, p$, are generalized $(F, \theta_{1i}, \rho_{1i}, d_{1i})$-convex at $\bar{x}$, $h_j, j = 1, \ldots, m$, are generalized $(F, \theta_{2j}, \rho_{2j}, d_{2j})$- convex at $\bar{x}$ and the inequality*

$$\sum_{i=1}^{p} \tau_i \bar{\rho}_i \frac{[\bar{d}_i(x,\bar{x})]^2}{\bar{\theta}_i(x,\bar{x})} + \sum_{j=1}^{m} \lambda_j \rho_{2j} \frac{[d_{2j}(x,\bar{x})]^2}{\theta_{2j}(x,\bar{x})} \geq 0$$

holds, where $\bar{\theta}_i(x,\bar{x}) = \theta_{1i}(x,\bar{x}) \frac{g_i(\bar{x})}{g_i(x)}$, $\bar{\rho}_i = \rho_{1i}\left(1 + \frac{f_i(\bar{x})}{g_i(\bar{x})}\right)$, *and* $\bar{d}_i(x,\bar{x}) = \frac{d_{1i}(x,\bar{x})}{[g_i(x)]^{1/2}}$, *then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is an efficient solution of* (MFD).

Proof Since $\bar{x}$ is an efficient solution for (MFP) and the Constraint Qualification A holds at $\bar{x}$ for (FP_k), $k = 1, \ldots, p$, from Theorem 2, there exist $\bar{\tau} \in R^p$, $\bar{\lambda} \in R^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a feasible solution of (MFD). If $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is not efficient for (MFD), then there must exist a feasible solution (x^*, τ^*, λ^*) of (MFD) such that

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} \leq \frac{f_i(x^*)}{g_i(x^*)}, \quad \text{for all} \quad i = 1, \ldots, p$$

and

$$\frac{f_k(\bar{x})}{g_k(\bar{x})} < \frac{f_k(x^*)}{g_k(x^*)}, \quad \text{for some} \quad k \in \{1, \ldots, p\},$$

which contradicts the weak duality result appearing in Theorem 4. Therefore, $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is an efficient solution of (MFD). The proof is complete.

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